Computer Vision (600.461/600.661)
Homework 1: Mathematical Background

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Due 09/11/2014, 11.59PM Eastern

1. **Properties of Symmetric Matrices.** Let $S \in \mathbb{R}^{n \times n}$ be a real symmetric matrix. Show that:
   (a) All the eigenvalues of S are real, i.e., $\sigma(S) \subset \mathbb{R}$.
   (b) Let (λ, v) be an eigenvalue-eigenvector pair. If $\lambda_i \neq \lambda_j$, then $v_i \perp v_j$; i.e., eigenvectors corresponding to distinct eigenvalues are orthogonal.
   (c) There always exist n orthonormal eigenvectors of S, which form a basis of $\mathbb{R}^n$.
   (d) S is positive definite (positive semidefinite) if and only if all of its eigenvalues are positive (non-negative), i.e., $S \succ 0$ ($S \succeq 0$), iff $\forall i = 1, 2, \ldots, n$, $\lambda_i > 0$ ($\lambda_i \geq 0$).
   (e) If $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ are the sorted eigenvalues of S, then $\max_{\|\boldsymbol{x}\|_2=1} \boldsymbol{x}^\top S \boldsymbol{x} = \lambda_1$ and $\min_{\|\boldsymbol{x}\|_2=1} \boldsymbol{x}^\top S \boldsymbol{x} = \lambda_n$.

2. **Properties of the SVD.** Let $A = U \Sigma V^\top$ be the SVD of a matrix $A \in \mathbb{R}^{m \times n}$ of rank r. Show that:
   (a) $A\boldsymbol{v}_j = \sigma_j \boldsymbol{u}_j$ for $j = 1, \ldots, r$ and $A^\top \boldsymbol{u}_j = \sigma \boldsymbol{v}_j$ for $j = 1, \ldots, r$.
   (b) The range or image of A is spanned by the left singular vectors of A associated with its nonzero singular values, i.e., $\operatorname{range}(A) = \operatorname{span}\{\boldsymbol{u}_i\}_{i=1}^r$.
   (c) The kernel or null space of A is spanned by the right singular vectors of A associated with its zero singular values, i.e., $\ker(A) = \operatorname{span}\{\boldsymbol{v}_i\}_{i=r+1}^m$.
   (d) The squared Frobenius norm of A is equal to the sum of the squared singular values of A, i.e., $\|A\|_F^2 = \sum_{ij} a_{ij}^2 = \sum_{k=1}^r \sigma_k^2$.
   (e) The right singular vector of A associated to its smallest singular value, $\boldsymbol{v}_m$, is a solution to the optimization problem $\min_{\boldsymbol{x}} \|A\boldsymbol{x}\|_2^2$ such that $\|\boldsymbol{x}\|_2 = 1$.

3. **Least Squares.** Recall that the pseudo inverse of a matrix $A \in \mathbb{R}^{m \times n}$ is the unique matrix $A^\dagger \in \mathbb{R}^{n \times m}$ such that: (i) $AA^\dagger A = A$, (ii) $A^\dagger A A^\dagger = A^\dagger$, (iii) $(AA^\dagger)^\top = AA^\dagger$, and (iv) $(A^\dagger A)^\top = A^\dagger A$. Let $A = U\Sigma V^\top$ be the SVD of A, let $r = \operatorname{rank}(A)$ and let $\boldsymbol{b} \in \mathbb{R}^m$. Show that:
   (a) The pseudo-inverse of A is given by $A^\dagger = V_r \Sigma_r^{-1} U_r^\top$, where $A = U_r \Sigma_r V_r^\top$ is the compact SVD of A.
   (b) $\boldsymbol{x}^* = A^\dagger \boldsymbol{b}$ is a solution to the optimization problem $\min_{\boldsymbol{x}} \|A\boldsymbol{x} - \boldsymbol{b}\|_2^2$. When is $\boldsymbol{x}^*$ the unique solution?
   (c) If $\boldsymbol{b} \in \operatorname{range}(A)$, $\boldsymbol{x}^* = A^\dagger \boldsymbol{b}$ is the solution to the optimization problem $\min_{\boldsymbol{x}} \|\boldsymbol{x}\|_2^2$ such that $A\boldsymbol{x} = \boldsymbol{b}$.

Submission instructions. Send email to vision14jhu@gmail.com with subject 600.461/600.661:HW1 and attachment `firstname-lastname-hw1-vision14.zip` or `firstname-lastname-hw1-vision14.tar.gz`. The attachment should have the following content:

1. A file called `hw1.pdf` containing your answers to each one of the analytical questions. If at all possible, you should generate this file using the latex template hw1-vision14.tex. If not possible, you may use another editor, or scan your handwritten solutions. But note that you must submit a single PDF file with all your answers.